# Development of Generalized Refinement Strategies in Composite Stationary Iterative Solvers for Linear Systems

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**Abstract:** Linear equations have many applications in natural sciences, engineering, business, social sciences, and medicine. However, solving these types of systems is an important challenge in Numerical Linear Algebra (NLA). There are two types of Methods to solve the systems: direct approaches and indirect approaches. Iterative methods are very successful in solving large, sparse linear problems. Iterative approaches often outperform direct methods, particularly when working with sparse coefficient matrices.

This research introduces the composite stationary iterative approach for solving linear systems of equations (GCST). And validate by comparing the proposed method with existing methods, in terms of spectral radius, no. of iterations, and convergence rate. Under specific conditions, proposed methods efficiently solve linear systems with coefficient matrices that are irreducibly diagonally dominant (IDD), strictly diagonally dominant (SDD), M-matrices, or symmetric positive definite. MATLAB (R2014b) was used to compute numerical tests.

**Keywords:** Iteration Process, Linear System, Strictly Diagonally Dominant, Composite Refinement, Irreducibly Diagonally Dominant, Symmetric Positive Definite, Rapid Convergence.

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## 1. Introduction

Solving linear systems of equations is one of the main goals of linear systems of equations (LSE). LSE's primary challenge is to resolve

$$A\eta = b \tag{1}$$

for $\eta$, where $A \in \mathbb{R}^{n\times n}$ is a given matrix, $b \in \mathbb{R}^n$ is a given vector, and $\eta \in \mathbb{R}^n$ is an unknown vector that is to be found. The solution of LSE can be found by using analytic or non-analytic methods, which could involve iterative methods that are powerful yet computationally friendly. In general, one should prefer an analytic method for solving an LSE due to a much higher level of accuracy in the solution, but in cases where the coefficient matrix $A$ is significantly large (and with a large number of zero elements), the computations with analytic methods are likely to become too complex and cumbersome. It will be in these cases advantageous to use an iterative method.

There is a plethora of iterative solvers present in the literature for solving LSEs, and they are still developing to this day. In the 18th century, Augustin Louis Cauchy developed the method of steepest descent. In the 19th century, Jacobi and Gauss-Seidel developed their respective methods. The method of the Conjugate Gradient was developed in the 20th century, and iterative solvers continue to be developed and improved. The modernization of technology and the increasing functionality of computers have led to the advancement of iterative solvers. Any newly developed iterative solver should be designed in a way that is efficient and scalable for computer applications. Alongside developing new techniques, many researchers have worked on improving the existing methods to improve their convergence and lower the required number of iterations to solve a problem. In the classical methods developed by Gauss-Seidel and Jacobi, the solution of an $n \times n$ linear system $A\,z = b$ starts with an initial approximation of the unknown vector, say $\eta^{(0)}$. It produces a series of vectors $\{\eta^{(k)}\}_{k=0}^{\infty}$ that converges to the solution vector $\eta$. For a general Jacobi method and Gauss-Seidel method, the iterative technique for solving LSEs involves a process that converts $A\,z = b$ into a comparable system of that kind $\eta = T\eta + c$ for some fixed matrix $T$ (called the iteration matrix) and a constant vector $c$ . After the initial vector $\eta^{(0)}$ is selected, the sequence of approximate solution vectors is generated by computing $\eta^{(k+1)} = T\eta^{(k)} + c$, for each $k = 0, 1, 2, \ldots$ [1-5].

These methods work well with systems that are not very large, and the rate at which the sequence $\{\eta^{(k)}\}_{k=0}^{\infty}$ will converge to the vector depends upon the spectral radius of the iteration matrix $T$ . Composite refinement is a technique [8,9] that can be used to improve the convergence of iterative methods. The basic idea is to combine two different iterative methods, each of which has its strengths. The complexity of the algorithm increases due to the composition, but the achieved speed in convergence due to refinement dominates the increase in computational costs that appears in every step. The refinement process is fundamentally based on the use of a virtual step $\eta^{(vir)}$, which resembles the approach used in double sweep methods as well as in symmetric and unsymmetric procedures, but without the reversal of the ordering equation [11].

The general iterative method will be,

$J\eta^{(k)} = K\eta^{(k-1)} + b$

rewrite as

$\eta^{(vir)} = J^{-1}K\eta^{(k-1)} + J^{-1}b,$

Further iterations make advantage of this virtual computed data, such as,

$\eta^{(k)} = J^{-1}K\eta^{(vir)} + J^{-1}b, k = 0, 1, 2, \ldots$

which can be rearranged in the form

$\eta^{(k)} = (J^{-1}K)^2\eta^{(k-1)} + (I + J^{-1}K)J^{-1}b, k = 0, 1, 2, \ldots$

In the composite refinement approach, different iterative techniques are employed in successive sweeps. This concept is applied to the three classical iterative solvers: the Gauss–Seidel method, the Jacobi method, and the SOR method [11, 16, 19]. In this research, a Generalized refinement of the Composite stationary iterative technique has been proposed. The proposed method increases its rate of convergence and minimizes the number of iterations compared to the refinement of composite methods. The convergence of the proposed method is guaranteed for some special matrices, such as strictly diagonally dominant (SDD), irreducibly diagonally dominant (IDD), symmetric positive definite (SPD), with some specific conditions and M-matrices [17].

***1.1 Generalized Methods*** [2,4,6,7,10-12].

### *1.1.1 Generalized Jacobi Method*

$$\eta^{(k)} = T_m^{-1}(E_m + F_m)\eta^{(k-1)} + T_m^{-1}b, \qquad k = 1,2,\dots \tag{2}$$

### *1.1.2 Generalized Gauss Seidel Method.*

$$\eta^{(k)} = (T_m - E_m)^{-1}F_m\eta^{(k-1)} + (T_m - E_m)^{-1}b \quad k = 1,2,\dots \tag{3}$$

### *1.1.3 Generalized Successive Over-Relaxation Method*

$$\eta^{(k)} = (T_m - wE_m)^{-1}[(1-w)T_m + wF_m]\eta^{(k-1)} + (T_m - wE_m)^{-1}wb \tag{4}$$

## 2. Derivation of Proposed Algorithm

The following equations (3) and (4) propose the Generalized Refinement of the Composite Stationary Iterative Technique for Linear Systems (GCST).

We have equation (3) as $\eta^{(k)} = (T_m - E_m)^{-1}F_m\eta^{(k-1)} + (T_m - E_m)^{-1}b$

and equation (4) as $\eta^{(k)} = (T_m - \omega E_m)^{-1}[(1-w)T_m + \omega F_m]\eta^{(k-1)} + (T_m - \omega E_m)^{-1}\omega b$

Using the Composite Refinement Technique, we obtain:

$$\eta^{(vir)} = (T_m - \omega E_m)^{-1}[(1-\omega)T_m + \omega F_m]\eta^{(k-1)} + (T_m - \omega E_m)^{-1}\omega b \tag{i}$$

$$\eta^{(k)} = (T_m - E_m)^{-1}F_m\eta^{(vir)} + (T_m - E_m)^{-1}b \tag{ii}$$

Substitute the value of equation (i) into equation (ii) and simplify,

$$\eta^{(k)} = (T_m - E_m)^{-1}F_m\{(T_m - \omega E_m)^{-1}[(1-\omega)T_m + \omega F_m]\eta^{(k-1)} + (T_m - \omega E_m)^{-1}\omega b\} + (T_m - E_m)^{-1}b$$

$\eta^{(k)} = (T_m - E_m)^{-1}F_m\,(T_m - \omega E_m)^{-1}[(1-\omega)T_m + \omega F_m]\eta^{(k-1)} + (T_m - E_m)^{-1}F_m(T_m - \omega E_m)^{-1}\omega b + (T_m - E_m)^{-1}b$

$$\eta^{(k)} = (T_m - E_m)^{-1}F_m\,(T_m - \omega E_m)^{-1}[(1-\omega)T_m + \omega F_m]\eta^{(k-1)} + (T_m - E_m)^{-1}b\{F_m(T_m - \omega E_m)^{-1}\omega + I\} \tag{5}$$

Equation (5) depicts the new proposed technique (GCST), which can be transformed into

$\eta^{(k)} = T_{GCST}\,\eta^{(k-1)} + C_{GCST} \qquad k = 1,2,3,\ldots$

where:

$T_{GCST} = (T_m - E_m)^{-1}F_m\,(T_m - \omega E_m)^{-1}[(1-\omega)T_m + \omega F_m]$

$C_{GCST} = (T_m - E_m)^{-1}b\{F_m(T_m - \omega E_m)^{-1}\omega + I\}$

## 3. Convergence Theory

**Definition 1**: "$G$ be a square matrix of order $n \times n$, where $G \in \mathbb{C}^{n\times n}$, is called reducible if $\exists$ $P$(a permutation matrix) and $k \in \{1,2,\ldots,n-1\}$ such that $PGP^T = \begin{pmatrix} g_{11} & g_{12} \\ 0 & g_{22} \end{pmatrix}$, where $G_{11}$is $k \times k$ and $G_{22}$ is $(n-k)\times(n-k)$. The matrix $G$ is said to be irreducible if it is not reducible" [13].

**Definition 2:** "$A$ matrix is said to be diagonally dominant (DD) if it satisfies the condition $|a_{i,i}| \geq \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{i,i}|, (1 \leq i \leq n)$, where the matrix is square of order n"[14].

**Definition 3**: "$A$ matrix is said to be strictly diagonally dominant (SDD) if it satisfies the condition $|a_{i,i}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{i,i}|, (1 \leq i \leq n)$, where the matrix is square of order n" [14].

**Definition 4**: "$A$ matrix is said to be irreducible diagonally dominant (IDD), If $0(A) = n \times n$ with $|a_{i,i}| \geq \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{i,i}|, (1 \leq i \leq n)$ and $|a_{k,k}| > \sum_{\substack{j=1 \\ j\neq k}}^{n} |a_{k,k}|$, for at least one $k$ for $k \in \{1,2,3,\ldots n\}$." [13], [14].

**Definition 5**: "$A$ matrix is said to be strictly diagonally dominant (SDD) if it satisfies the condition $A = A^T$ and $u^T A u > 0 \ \forall\ a \neq 0$, where the matrix is a square of order n" [14].

Note: In SPD, every $a_{ii} > 0$ for $i = 1, 2, \ldots, n$.

**Definition 6**: "If $A$ is an $n \times n$ square matrix with eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$, the spectral radius is denoted by $\rho(A)$, which is defined as: $\rho(A) = max\{|\lambda_i|, for\ i = 1,2, \ldots n\}$" [13].

**Definition 7:** “Definition 7: Let $A = (a_{i\,j}) \in R^{n\times n}$, and $A = J - K$. If $\det(J)$ does not equal zero, then $A$ is split.

a) If $\rho(J^{-1}\,K) < 1$, it is considered convergent.
b) If $J^{-1} \geq 0$ and $K \geq 0$, the function is considered regular.
c) If $J^{-1}K \geq 0$, it is non-negative.
a) If $det(J) \neq 0$ and $K \geq 0$, it is considered M-splitting” [13].

**Definition 8**: “The convergence rate of an iterative method is $R(T) = -\log 10\ [\rho(T)]$ or $\tau = -\ln\rho$. This convergence rate is also known as the asymptotic rate of convergence.” [13].

**Definition 9**: ‘’Consider a function $||\,.\,||: R^n \to R$, that has the following characteristics:

$\forall\ \mu, \nu \in R^n$ and $a \in R$

a) $||\mu|| \geq 0$, non-negativity
b) $||\mu|| = 0$ if $u = 0$, null vector
c) $||a\mu|| = |a|\ ||\mu||$, it is non-negative scalar multiple
d) $||\mu + \nu|| \leq ||\mu||$+$||\nu||$, the norm's sum is less than the sum of its parts

$\|\mu\|_2 = \{\sum_i^n \mu_i^2\}^{\frac{1}{2}}$ is the Euclidean norm $\|\mu\| = \max_{1\leq i\leq n}|\mu_i|$ ([15], p.432) is the Infinity norm.

**Theorem 1:** The GSOR technique for SDD-matrices converges if $0 < \omega < 2$ [16].

**Theorem 2:** The GCST technique is convergent for any initial guess $\eta^{(0)}$ if A is an SDD matrix.

**Proof.**

We know that if $A\eta = b$, $\eta$ is the precise solution to the linear system of equations. Given that A is a strictly diagonally dominant matrix and GSOR,

$$\eta^{vir} = (T_m - wE_m)^{-1}[(1-w)T_m - wF_m]\eta^{(n)} + (T_m - wE_m)^{-1}wb$$

Now using Theorem 7, GSOR converges,

So, $\eta^{vir} \to \eta$

We can write GCST as,

$$\eta^{(n+1)} = \eta^{vir} + (T_m - E_m)^{-1}(b - A\,\eta^{vir})$$

$\eta^{(n+1)} - \eta = \eta^{vir} - \eta + (T_m - E_m)^{-1}(b - A\ \eta^{vir})$

Taking norms on both sides,

$\|\eta^{(n+1)} - \eta\| = \|\eta^{vir} - \eta + (T_m - E_m)^{-1}(b - A\ \eta^{vir})\|$

$\|\eta^{(n+1)} - \eta\| \le \|\eta^{vir} - \eta\| + \|(T_m - E_m)^{-1}\| \|(b - A\ \eta^{vir})\|$

$\|\eta^{(n+1)} - \eta\| \le \|\eta - \eta\| + \|(T_m - E_m)^{-1}\| \|(b - b)\|$

$\|\eta^{(n+1)} - \eta\| \le 0 + \|(T_m - E_m)^{-1}\| \times 0$

$\|\eta^{(n+1)} - \eta\| \le 0$

Consequently,

$\|\eta^{(n+1)} - \eta\| \rightarrow 0$

Then, by using Theorem 1

$\rho((T_m - E_m)^{-1}F_m(T_m - wE)^{-1}[(1-w)T_m + wF]) < 1$

Consequently, GCST is convergent for any strictly diagonally dominating matrix..

**Theorem 3:** Let A be a matrix of order $n$, where $m < n$ and m belongs to a set of natural numbers $\mathbb{N}$, then the Generalized Conjugate Splitting Technique (GCST) is guaranteed to converge for any vector $\eta^{(0)}$.

**Proof.**

It is known that for the linear system $A\eta = b$, vector $\eta$ represents the exact solution. Given that matrix $A$ is strictly diagonally dominant (SDD) and that the GSOR method is employed, the convergence of the GCST method follows.

$\eta^{vir} = (T_m - wE_m)^{-1}[(1-w)T_m - wF_m]\eta^{(n)} + (T_m - wE_m)^{-1}wb$

Now using Theorem 1, GSOR converges,

So, $\eta^{vir} \rightarrow \eta$

We can write GCST as,

$\eta^{(n+1)} = \eta^{vir} + (T_m - E_m)^{-1}(b - A\ \eta^{vir})$

$\eta^{(n+1)} - \eta = \eta^{vir} - \eta + (T_m - E_m)^{-1}(b - A\ \eta^{vir})$

Taking norms on both sides,

$$\left\|\eta^{(n+1)} - \eta\right\| = \left\| \eta^{vir} - \eta + (T_m - E_m)^{-1}(b - A\ \eta^{vir})\right\|$$

$$\left\| \eta^{(n+1)} - \eta\right\| \leq \left\| \eta^{vir} - \eta\right\| + \| (T_m - E_m)^{-1}\| \left\|(b - A\ \eta^{vir})\right\|$$

$$\left\| \eta^{(n+1)} - \eta\right\| \leq \|\eta - \eta\| + \| (T_m - E_m)^{-1}\| \|(b - b)\|$$

$$\left\| \eta^{(n+1)} - \eta\right\| \leq 0 + \| (T_m - E_m)^{-1}\| \times 0$$

$$\left\| \eta^{(n+1)} - \eta\right\| \leq 0$$

Consequently,

$$\left\| \eta^{(n+1)} - \eta\right\| \to 0$$

Then, by using Theorem 1

$$\rho((T_m - E_m)^{-1}F_m(T_m - wE)^{-1}[(1-w)T_m + wF]) < 1$$

GCST converges for any vector $\eta^{(0)}$.

**Theorem 4:** If GSOR converges to $\eta^{(0)}$, then GCST converges faster than GSOR and RGSSOR.

**Proof:**

Let, $W_m = (T_m - wE_m)^{-1}[(1-w)T_m + wF_m]$

$$Y_m = (T_m - wE_m)^{-1}wb$$

$$P_m = (D - L)^{-1}U(D - wL)^{-1}[(1-w)D + wU]$$

$$Q_m = (D - L)^{-1}[I + U(D - wL)^{-1}w]b$$

$$R_m = (T_m - E_m)^{-1}F_m(T_m - wE_m)^{-1}[(1-w)T_m + wF_m]$$

$$S_m = (T_m - E_m)^{-1}[I + F_m(T_m - wE_m)^{-1}w]b$$

Then we can write the equations of GSOR, RGSSOR, and GCST as,

$$\eta^{(n+1)} = W_m\eta^{(n)} + Y_m$$

$$\eta^{(n+1)} = P_m\eta^{(n)} + Q_m$$

$\eta^{(n+1)} = R_m\eta^{(n)} + S_m$ respectively.

GSOR converges means $\|W_m\| < 1$. We know that for $A\eta = b$, $\eta$ is the solution.

Let's take GSOR first, we have,

$$\eta^{(n+1)} = W_m\eta^{(n)} + Y_m$$

$$\eta^{(n+1)} - z = W_m\eta^{(n)} - \eta + Y_m$$

Adding and subtracting $W_m\eta$ on R.H.S keeping in view that $z = W_m\eta^{(n)} + Y_m\eta^{(n+1)} - \eta = W_m z^{(n)} - W_m\eta + W_m\eta + Y_m - \eta$

$$\eta^{(n+1)} - \eta = W_m(\eta^{(n)} - \eta) + \eta - \eta$$

$$\eta^{(n+1)} - \eta = W_m(\eta^{(n)} - \eta)$$

Taking norm on both sides

$$\left\|\eta^{(n+1)} - \eta\right\| = \left\|W_m(\eta^{(n)} - \eta)\right\|$$

$$\left\|\eta^{(n+1)} - \eta\right\| \leq \|W_m\|\left\|(\eta^{(n)} - \eta)\right\| \tag{a}$$

Now for RGSSOR,

$$\eta^{(n+1)} = P_m\eta^{(n)} + Q_m$$

$$\eta^{(n+1)} - \eta = P_m\eta^{(n)} - z + Q_m$$

Adding and subtracting $P_m\eta$ on R.H.S keeping in view that $\eta = P_m\eta^{(n)} + Q_m$

$$\eta^{(n+1)} - \eta = P_m\eta^{(n)} - P_m\eta + P_m\eta + Q_m - \eta$$

$$\eta^{(n+1)} - \eta = P_m(\eta^{(n)} - \eta) + \eta - \eta$$

$$\eta^{(n+1)} - \eta = P_m(\eta^{(n)} - \eta)$$

Taking norm on both sides

$$\left\|\eta^{(n+1)} - \eta\right\| = \left\|P_m(\eta^{(n)} - \eta)\right\|$$

$$\left\|\eta^{(n+1)} - \eta\right\| \leq \|P_m\|\left\|(\eta^{(n)} - \eta)\right\| \tag{b}$$

Now let us consider GCST,

$$\eta^{(n+1)} = R_m\eta^{(n)} + S_m$$

$\eta^{(n+1)} - \eta = R_m\eta^{(n)} - \eta + S_m$

Adding and subtracting $R_m\eta$ on R.H.S keeping in view that $\eta = R_m\eta^{(n)} + S_m\eta^{(n+1)} - \eta = R_m\eta^{(n)} - R_m\eta + R_m\eta + S_m - \eta$

$\eta^{(n+1)} - \eta = R_m(\eta^{(n)} - \eta) + \eta - \eta$

$\eta^{(n+1)} - \eta = R_m(\eta^{(n)} - \eta)$

Taking norm on both sides

$\left\|\eta^{(n+1)} - \eta\right\| = \left\|R_m(\eta^{(n)} - \eta)\right\|$

$$\left\|\eta^{(n+1)} - \eta\right\| \leq \|R_m\|\left\|(\eta^{(n)} - \eta)\right\| \qquad \text{(c)}$$

From equations.

(a), (b) and (c) we have,

$\|R_m\| < \|P_m\| < \|W_m\| \qquad Since\ \|W_m\| < 1$

Hence, the GCST method converges faster than GSOR and RGSSOR.

## 4. Numerical Experiments and Discussions

This study examines many forms of linear equation systems, including IDD, SPD, SDD, and M-matrix systems. The suggested algorithms' usefulness is proved by comparing them to the Composite RJSOR, Composite RSORJ, Composite RGSSOR, and Composite RSORGS approaches. Comparative tables are used to analyze critical characteristics such as the number of iterations, spectral radius, and rate of convergence.

### *4.1 Numerical Experiments*

**Problem 1: M-MATRIX** [4, 18]

$$A = \begin{bmatrix} 6 & -9 & 0 & 0 & 0 & 0 \\ -3 & 12 & -3 & 0 & -3 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & 6 & -9 & 0 \\ 0 & -3 & 0 & -3 & 12 & -3 \\ 0 & 0 & -3 & 0 & -3 & 12 \end{bmatrix}, \quad \eta = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \\ \eta_6 \end{bmatrix} \quad b = \begin{bmatrix} -5 \\ 2 \\ 3 \\ -4 \\ -1 \\ 5 \end{bmatrix}$$

**Table 1. Comparison Table of Problem 1: GCST vs. Counterpart Methods**

| Methods | RJSOR | RSORJ | RGSSOR | RSORGS | GCST |
|---|---|---|---|---|---|
| No of Iter. | 20 | 20 | 14 | 13 | 4 |
| Spectral Rad. | 0.5894 | 0.5894 | 0.4015 | 0.4015 | 0.0239 |
| Rate of Conv. | 0.2296 | 0.2296 | 0.3963 | 0.3963 | 1.6216 |

**Problem 2: IDD MATRIX [8].**

$$A = \begin{bmatrix} 4 & 1 & 2 & -1 \\ 3 & 6 & -1 & 2 \\ 2 & -1 & 5 & -3 \\ 4 & 1 & -3 & -8 \end{bmatrix}, \quad \eta = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 2 \end{bmatrix}$$

**Table 2. Comparison table of Problem 2: GCST vs. Counterpart Methods.**

| Methods | RJSOR | RSORJ | RGSSOR | RSORGS | GCST |
|---|---|---|---|---|---|
| No. of Iter. | 14 | 14 | 11 | 8 | 5 |
| Spectral Rad. | 0.4015 | 0.4015 | 0.2813 | 0.2813 | 0.046 |
| Rate of Conv. | 0.3936 | 0.3936 | 0.5508 | 0.5508 | 1.3372 |

**Problem 3: SDD AND M-MATRIX [4].**

$$A = \begin{bmatrix} 5 & -1 & 0 & 0 & -1 & 0 & 0 & -1 \\ -1 & 5 & -1 & 0 & 0 & 0 & -1 & -1 \\ 0 & -1 & 5 & -1 & 0 & -1 & -1 & 0 \\ 1 & 0 & -1 & 5 & -1 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 & 5 & -1 & 0 & -1 \\ 0 & 0 & -1 & -1 & 0 & 5 & 1 & -1 \\ -1 & 0 & 0 & 0 & -1 & 0 & 5 & -1 \\ -1 & 0 & -1 & 0 & -1 & 0 & -1 & 5 \end{bmatrix} \quad \eta = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \\ \eta_6 \\ \eta_7 \\ \eta_8 \end{bmatrix} \quad b = \begin{bmatrix} -2 \\ -1 \\ 4 \\ 13 \\ 4 \\ 2 \\ 9 \\ 12 \end{bmatrix}$$

**Table 3. Comparison table of Problem 3: GCST vs. Counterpart Methods.**

| Methods | RJSOR | RSORJ | RGSSOR | RSORGS | GCST |
|---|---|---|---|---|---|
| No of Iter. | 9 | 9 | 6 | 6 | 4 |
| Spectral Rad. | 0.1799 | 0.1799 | 0.0352 | 0.0352 | 0.0138 |
| Rate of Conv. | 0.745 | 0.745 | 1.4535 | 1.4535 | 1.8601 |

**Problem 4: SDD MATRIX [11].**

$$A = \begin{bmatrix} 1 & -0.25 & 0.25 & 0 \\ 0.25 & 1 & 0.125 & -0.375 \\ -0.25 & 0 & 1 & -0.5 \\ 0 & -0.125 & 0.125 & 1 \end{bmatrix} \quad \eta = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 1 \\ 0.25 \\ 1 \end{bmatrix}$$

**Table 4. Comparison table of Problem 4: GCST vs. Counterpart Methods.**

| Methods | RJSOR | RSORJ | RGSSOR | RSORGS | GCST |
|---|---|---|---|---|---|
| No of Iter. | 707 | 6 | 5 | 5 | 3 |
| Spectral Rad. | 0.1153.1153 | 0.1153 | 0.0509 | 0.0509 | 0.01 |
| Rate of Conv. | 0. 0.93809380 | 0.938 | 1.2933 | 1.2933 | 2 |

**Problem 5: SDD and SPD MATRIX [15].**

$$A = \begin{bmatrix} 4 & 1 & -1 & 0 \\ 1 & 3 & -1 & 0 \\ -1 & -1 & 5 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix} \qquad \eta = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix} \qquad b = \begin{bmatrix} 7 \\ 8 \\ -4 \\ 6 \end{bmatrix}$$

**Table 5. Comparison table of Problem 5: GCST vs. Counterpart Methods.**

| Methods | RJSOR | RSORJ | RGSSOR | RSORGS | GCST |
|---|---|---|---|---|---|
| No of Iter. | 10 | 11 | 7 | 7 | 4 |
| Spectral Rad. | 0.2675 | 0.2675 | 0.103 | 0.103 | 0.012 |
| Rate of Conv. | 0.5727 | 0.5727 | 0.9872 | 0.9872 | 1.9208 |

Tables 1 to 5 present the solutions of linear systems using various methods: RJSOR, RSORJ, RGSSOR, RSORGS, GCST 1, and GCST 2. These tables provide a comparative analysis of these methods based on the number of iterations, spectral radius, and rate of convergence. Notably, the GCST method demonstrates superior performance by requiring fewer iterations, achieving a lower spectral radius, and exhibiting a faster rate of convergence compared to the other methods. In these problems, the parameter ω is set to its optimal value, and the bandwidth is chosen as $m = 1$.

## 5. Conclusions

The proposed numerical technique, known as the "Generalized Composite Stationary Iterative Technique for Solving Systems of Linear Equations," significantly reduces the number of iterations required, decreases the spectral radius, and improves the convergence rate compared to the RJSOR, RSORJ, RGSSOR, and RSORGS methods. The comparison Tables demonstrate that GCST achieves rapid convergence and higher accuracy, especially when optimal parameter choices (ω or omega) are applied.

In conclusion, the "Generalized Refinement of Composite Stationary Iterative Technique for Solving Systems of Linear Equations" consistently outperforms the methods in most cases, offering a more efficient and accurate solution.

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